5. For ALL APPLICANTS.
The Millennium school has 1000 students and 1000 student lockers. The lockers are in a line in a long corridor and are numbered from 1 to 1000.

Initially all the lockers are closed (but unlocked).

The first student walks along the corridor and opens every locker.

The second student then walks along the corridor and closes every second locker, i.e. closes lockers 2, 4, 6, etc. At that point there are 500 lockers that are open and 500 that are closed.

The third student then walks along the corridor, changing the state of every third locker. Thus s/he closes locker 3 (which had been left open by the first student), opens locker 6 (closed by the second student), closes locker 9, etc.

All the remaining students now walk by in order, with the n^{th} student changing the state of every n^{th} locker, and this continues until all 1000 students have walked along the corridor.

(i) How many lockers are closed immediately after the third student has walked along the corridor? Explain your reasoning.

(ii) How many lockers are closed immediately after the fourth student has walked along the corridor? Explain your reasoning.

(iii) At the end (after all 1000 students have passed), what is the state of locker 100? Explain your reasoning.

(iv) After the hundredth student has walked along the corridor, what is the state of locker 1000? Explain your reasoning.

6.
For APPLICANTS IN COMPUTER SCIENCE and MATHEMATICS & COMPUTER SCIENCE ONLY.

(i) A, B and C are three people. One of them is a liar who always tells lies, another is a saint who always tells the truth, and the third is a switcher who sometimes tells the truth and sometimes lies. They make the following statements:
A: I am the liar.
B: A is the liar.
C: I am not the liar.
Who is the liar among A, B and C? Briefly explain your answer.

(ii) P, Q and R are three more people, one a liar, one a saint, and the third a *contrarian* who tells a lie if he is the first to speak or if the preceding speaker told the truth, but otherwise tells the truth. They make the following statements:
P: Q is the liar.
Q: R is the liar.
R: P is the liar.
Who is the liar among P, Q and R? Briefly explain your answer.

(iii) X, Y and Z are three more people, one a liar, one a switcher and one a contrarian. They make the following statements:
X: Y is the liar.
Y: Z is the liar.
Z: X is the liar.
X: Y is the liar.
Y: X is the liar.
Who is the liar among X, Y and Z? Briefly explain your answer.

7.
For APPLICANTS IN COMPUTER SCIENCE ONLY.

Suppose you have an unlimited supply of black and white pebbles. There are four ways in which you can put two of them in a row: BB, BW , W B and WW.

(i) Write down the eight different ways in which you can put three pebbles in a row.

(ii) In how many different ways can you put N pebbles in a row?

Suppose now that you are not allowed to put black pebbles next to each other. There are now only three ways of putting two pebbles in a row, because BB is forbidden.

(iii) Write down the five different ways that are still allowed for three pebbles.
Now let r_N be the number of possible arrangements for N pebbles in a row, still under the restriction that black pebbles may not be next to each other, so $r_2 = 3$ and $r_3 = 5$.

(iv) Show that for $N \geq 4$ we have $r_N = r_{N-1} + r_{N-2}$. Hint: consider separately the case where the last pebble is white, and the case where it is black.

Finally, suppose that we impose the further restriction that the first pebble and the last pebble cannot both be black. Let w_N be the number of such arrangements for N pebbles; for example, $w_3 = 4$, since the configuration BW B is now forbidden.
(v) For $N \geq 5$, write down a formula for w_N in terms of the numbers r_i, and explain why it is correct.